

# RADFORD'S $S^4$ FORMULA FOR CO-FROBENIUS HOPF ALGEBRAS

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**ABSTRACT.** This note extends Radford's formula for the fourth power of the antipode of a finite dimensional Hopf algebra to co-Frobenius Hopf algebras and studies equivalent conditions to a Hopf algebra being involutory for finite dimensional and co-Frobenius Hopf algebras.

## 1. INTRODUCTION AND PRELIMINARIES

A key result in the theory of finite dimensional Hopf algebras is Radford's  $S^4$  formula. It asserts that for  $S$  the antipode of a finite dimensional Hopf algebra  $H$  over a field  $k$ , then for all  $h \in H$ ,

$$(1.1) \quad S^4(h) = g(\alpha \rightharpoonup h \leftharpoonup \alpha^{-1})g^{-1},$$

where  $g$  and  $\alpha$  are the distinguished grouplike elements of  $H$  and  $H^*$ , respectively. This formula was initially proved by Larson in [15, Theorem 5.5] for finite dimensional unimodular Hopf algebras and extended by Radford in [21, Proposition 6] to any finite dimensional Hopf algebra  $H$ . The techniques used by Larson and Radford are similar, having their roots in a paper of Sweedler [27]. Their idea was to compute the square, and then the fourth power, of the antipode in terms of a non-singular bilinear form associated to  $H$ . More recently, Schneider in [26, Theorem 3.8] used the fact that a finite dimensional Hopf algebra is a Frobenius algebra to provide a more transparent proof. The dual bases of  $H$  in terms of integrals for  $H$  and  $H^*$  were explicitly computed in two different ways. Then Radford's  $S^4$  formula follows from the two expressions for the Nakayama automorphism of  $H$ . Kadison and Stolin in [13] use this same point of view to obtain an analogous result for Hopf algebras over commutative rings that are Frobenius algebras as do Doi and Takeuchi [9] in their proof of the  $S^4$  formula for biFrobenius algebras. Similarly, Montgomery [18] uses this approach in her discussion of finite-dimensional Hopf algebras and new proof that in characteristic 0, a finite-dimensional Hopf algebra  $H$  is involutory if and only if  $H$  or  $H^*$  is semisimple.

Moreover, the  $S^4$  formula was recently generalized for quasi-Hopf algebras in [12, Corollary 6.3], for weak Hopf algebras in [20, Theorem 5.13] and for  $QFH$ -algebras (quasi-Hopf algebras over commutative rings that are Frobenius algebras) in [14,

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Theorem 3.3]. A braided version of the  $S^4$  formula is proved [4] and in [9]. Perhaps the most striking generalization is the categorical  $S^4$  formula associated to any finite tensor category, see [10, Theorem 3.3].

We begin with a review of the spaces of integrals in Hopf algebras and give a short proof of the  $S^4$  formula for finite dimensional Hopf algebras using integrals. We then prove an analogue of the  $S^4$  formula for co-Frobenius Hopf algebras, providing another new proof for the finite dimensional case. In the last section, we study conditions on left or right integrals which are equivalent to the Hopf algebra being involutory.

Key to our arguments is the use of the one dimensional spaces of integrals in  $H$  and  $H^*$  and versions of the isomorphism induced by the structure theorem for Hopf modules [28, Theorem 4.1.1]. We recall some basics about spaces of integrals.

**1.1. Preliminaries: Spaces of Integrals.** Throughout,  $H$  will be either a finite dimensional or a co-Frobenius Hopf algebra with antipode  $S$ . In either case,  $S$  is bijective with composition inverse denoted by  $S^{-1}$ . We will use the Heyneman-Sweedler notation [28],  $\Delta(h) = h_1 \otimes h_2$ ,  $h \in H$ , for comultiplication (summation understood). We assume that the reader is familiar with the basic theory of Hopf algebras; see [7, 19, 28] for example. We work over a commutative field  $k$  throughout.

Any Hopf algebra  $H$  is an  $H^*$ -bimodule where for  $h^* \in H^*$  and  $h \in H$ ,

$$(1.2) \quad h^* \rightharpoonup h = h^*(h_2)h_1, \text{ and } h \leftharpoonup h^* = h^*(h_1)h_2.$$

Also the algebra  $H^*$  is an  $H$ -bimodule, where for  $h^* \in H^*$  and  $h, h' \in H$ ,

$$(1.3) \quad \langle h \rightharpoonup h^*, h' \rangle = \langle h^*, h'h \rangle, \text{ and } \langle h^* \leftharpoonup h, h' \rangle = \langle h^*, hh' \rangle.$$

The fact that  $S$  is an anti-algebra morphism yields that for  $h \in H$ ,  $h^* \in H^*$ ,

$$(1.4) \quad (h \rightharpoonup h^*) \circ S = (h^* \circ S) \leftharpoonup S^{-1}(h), \text{ and } (h^* \leftharpoonup h) \circ S = S^{-1}(h) \rightharpoonup (h^* \circ S).$$

Now suppose that  $H$  is finite dimensional and recall the definition of an integral in  $H$ . An element  $t \in H$  is called a left integral in  $H$  if  $ht = \varepsilon(h)t$  for all  $h \in H$ . The space of left integrals is denoted  $\int_l^H$  and is a one dimensional ideal of  $H$ . Similarly if  $T \in H$  has the property that  $Th = \varepsilon(h)T$  for all  $h \in H$ , then  $T$  is called a right integral in  $H$ , and we write  $T \in \int_r^H$ .

Since  $\int_l^H$  is a one dimensional ideal in  $H$ , there exists a unique  $\alpha \in H^*$  such that for any  $t \in \int_l^H$  and for all  $h \in H$

$$(1.5) \quad th = \alpha(h)t.$$

The map  $\alpha$  is grouplike in  $H^*$ , the dual Hopf algebra of  $H$ , i.e.,  $\alpha$  is an invertible algebra map from  $H$  to  $k$ . Following Radford [22], we call  $\alpha$  the distinguished grouplike element of  $H^*$ . Clearly,  $\int_r^H = \int_l^H$  if and only if  $\alpha = \varepsilon$ . An integral is called cocommutative if it is a cocommutative element of  $H$ .

If  $0 \neq l$  is a left or a right integral in  $H$ , then the maps from  $H^*$  to  $H$  given by

$$(1.6) \quad h^* \mapsto (h^* \rightharpoonup l = h^*(l_2)l_1) \text{ and } h^* \mapsto (l \leftharpoonup h^* = h^*(l_1)l_2)$$

are bijections (see [7, 28]).

Now let  $H$  be a Hopf algebra which is not necessarily finite dimensional. An element  $\Lambda \in H^*$  is called a right integral for  $H$  in  $H^*$  if  $\Lambda(h_1)h_2 = \Lambda(h)1$  for all  $h \in H$ . The space of right integrals for  $H$  in  $H^*$  is denoted  $\int_r^{H^*}$ . Similarly,  $\lambda \in H^*$

is called a left integral for  $H$  in  $H^*$  if  $h_1\lambda(h_2) = \lambda(h)1$  for all  $h \in H$ ; the space of left integrals is denoted  $\int_l^{H^*}$ . The dimensions of the spaces  $\int_l^{H^*}$  and  $\int_r^{H^*}$  are equal and at most 1. For  $H$  finite dimensional, these are just the one dimensional spaces of integrals for  $H^*$ . An integral  $\Gamma$  in  $H^*$  for  $H$  is called cocommutative if  $\Gamma(hh') = \Gamma(h'h)$  for all  $h, h' \in H$ ; if some integral  $\Gamma$  is cocommutative, then all integrals are cocommutative.

Recall that a Hopf algebra  $H$  is called co-Frobenius if  $H^{*\text{rat}}$ , the unique maximal rational submodule of  $H^*$ , is nonzero, or equivalently if  $\int_l^{H^*} \neq 0$  or if  $\int_r^{H^*} \neq 0$ .

If  $0 \neq \Gamma$  is either a left or a right integral for  $H$  in  $H^*$ , then there are bijections from  $H$  to  $H^{*\text{rat}}$  given by

$$(1.7) \quad h \mapsto (h \rightharpoonup \Gamma) \in H^{*\text{rat}} \text{ and } h \mapsto (\Gamma \leftharpoonup h) \in H^{*\text{rat}}.$$

For  $H$  co-Frobenius, a grouplike element in  $H$  was defined in [3] as follows. Let  $0 \neq \lambda \in \int_l^{H^*}$ . If  $a \in H$  such that  $\lambda(a) = 1$  then by [3, Proposition 1.3] the element  $g^{-1} := a \leftharpoonup \lambda = \lambda(a_1)a_2$  is a grouplike element in  $H$  such that for all  $\lambda' \in \int_l^{H^*}$ ,  $\Lambda \in \int_r^{H^*}$ ,  $h \in H$ , we have that

$$(1.8) \quad \lambda'(h_1)h_2 = \lambda'(h)g^{-1} \text{ and } h_1\Lambda(h_2) = \Lambda(h)g.$$

For the sake of consistency with Radford's definition in the finite dimensional case where the distinguished grouplike in  $H$  was constructed using  $\Lambda \in \int_r^{H^*}$ , we call  $g \in H$  above the distinguished grouplike in  $H$ .

As well, the following relations hold [3]:

$$(1.9) \quad \lambda \circ S = g^{-1} \rightharpoonup \lambda, \lambda \circ S^{-1} = \lambda \leftharpoonup g^{-1} \text{ and } \lambda \circ S^2 = g^{-1} \rightharpoonup \lambda \leftharpoonup g.$$

The grouplike  $g = 1$  if and only if  $\int_r^{H^*} = \int_l^{H^*}$ .

**1.2. A proof of the  $S^4$  formula using integrals.** There are several short proofs of the formula for  $S^4$  in the literature, for example [24, p.596] gives a proof using the trace, Schneider [26] uses an approach with Frobenius algebras and [18, Theorem 2.10] gives a proof combining the Frobenius algebra approach of Schneider and that of Kadison and Stolin. As well, Radford [24] indicates that a short proof follows from [24, Theorem 3 (a),(b)]. We supply a proof for Radford's  $S^4$  formula, based on integrals.

Although the results in the next lemma are known (see [19] or [18] for (i), [24, Theorem 3 (a)] for an equivalent form of (ii), and [24, Theorem 3 (d)] for (iii)), we supply new proofs for (ii) and (iii).

**Lemma 1.1.** *Let  $H$  be finite dimensional,  $0 \neq t \in \int_l^H$ , and  $\alpha$  and  $g$  the distinguished grouplike elements of  $H^*$  and  $H$ , respectively. Let  $0 \neq \Lambda$  be the unique element of  $H^*$  such that  $t \leftharpoonup \Lambda = \Lambda(t_1)t_2 = 1$ . Then the following statements hold:*

- i)  $\Lambda \in \int_r^{H^*}$ ;
- ii) For any  $h \in H$ ,  $h^* \in H^*$  with  $h = t \leftharpoonup h^*$ , then  $h^* = \Lambda \leftharpoonup S(h) = \alpha(h_2)S^{-1}(h_1) \rightharpoonup \Lambda$ ;
- iii)  $\Delta(t) = S^2(t_2)g \otimes t_1$ .

*Proof.* i) The proof of this statement can be found in [19] or in [18, 2.5(i)].

ii) Since  $t \leftharpoonup \Lambda = 1$ , we have  $h(t \leftharpoonup \Lambda) = h = (t \leftharpoonup \Lambda)h$ . Then

$$h = h\Lambda(t_1)t_2 = \Lambda(S(h_1)h_2t_1)h_3t_2 = \Lambda(S(h)t_1)t_2 = t \leftarrow (\Lambda \leftarrow S(h)).$$

On the other hand, from (1.5) we have that  $h$  equals

$$\Lambda(t_1)t_2h = \Lambda(t_1h_2S^{-1}(h_1))t_2h_3 = \alpha(h_2)\Lambda(t_1S^{-1}(h_1))t_2 = t \leftarrow (\alpha(h_2)S^{-1}(h_1) \rightarrow \Lambda).$$

iii) Let  $h^* \in H^*$  with  $h^* = \Lambda \leftarrow h$ . By (1.8), and  $\Lambda(t_1)t_2 = \Lambda(t) = 1$ , we have  $S(h)g = S(h)\Lambda(t)g = S(h)(\Lambda(t_1)t_2)g = \Lambda(t_1)S^2(S^{-1}(h)t_2)g$ . Now

$$S(h)\Lambda(t)g = S(h)\Lambda(t_2)t_1 = \Lambda(h_3t_2)S(h_1)h_2t_1 = \langle \Lambda \leftarrow h, t_2 \rangle t_1 = h^*(t_2)t_1;$$

while

$$\Lambda(t_1)S^2(S^{-1}(h)t_2)g = \Lambda(h_1t_1)S^2(S^{-1}(h_3)h_2t_2)g = \Lambda(ht_1)S^2(t_2)g = h^*(t_1)S^2(t_2)g.$$

Thus  $h^*(t_2)t_1 = h^*(t_1)S^2(t_2)g$ , for all  $h^* \in H^*$ , and the statement follows.  $\square$

Now we give a short proof of the  $S^4$  formula using integrals.

**Theorem 1.2.** ([21, Proposition 6]) *Let  $H$  be a finite dimensional Hopf algebra with antipode  $S$ , and  $g$  and  $\alpha$  the distinguished grouplike elements of  $H$  and  $H^*$ , respectively. Then  $S^4(h) = g(\alpha \rightarrow h \leftarrow \alpha^{-1})g^{-1}$ , for any  $h \in H$ .*

*Proof.* Let  $h \in H$  and let  $h^* \in H^*$  be such that  $h = t \leftarrow h^* = h^*(t_1)t_2$ . For  $\Lambda \in \int_r^{H^*}$  as in Lemma 1.1, using Lemma 1.1 ii) and iii), then we have:

$$S^2(h)g = h^*(t_1)S^2(t_2)g = h^*(t_2)t_1 = \alpha(h_2)\Lambda(t_2S^{-1}(h_1))t_1.$$

Replacing  $h$  with  $S^2(h)$ , and using  $\alpha \circ S^2 = \alpha$ , (1.5), (1.8), and Lemma 1.1 i), we obtain

$$\begin{aligned} S^4(h)g &= (\alpha \circ S^2)(h_2)\Lambda(t_2S(h_1))t_1 = \alpha(h_4)\Lambda(t_2S(h_1))t_1S(h_2)h_3 \\ &= \alpha(h_3)\Lambda(t_2)t_1h_2(\alpha \circ S)(h_1) = \alpha(h_3)gh_2\alpha^{-1}(h_1) = g(\alpha \rightarrow h \leftarrow \alpha^{-1}), \end{aligned}$$

and the statement follows.  $\square$

## 2. RADFORD'S $S^4$ FORMULA FOR CO-FROBENIUS HOPF ALGEBRAS

Throughout this section,  $H$  will denote a co-Frobenius Hopf algebra, not necessarily finite dimensional. The following definition will be key in the computations that follow.

**Definition 2.1.** Let  $0 \neq \lambda \in \int_l^{H^*}$ . Let  $\chi : H \rightarrow H$  be the algebra isomorphism defined by

$$(2.1) \quad h \rightarrow \lambda = \lambda \leftarrow \chi(h), \quad \forall h \in H.$$

Following the notation for finite dimensional Frobenius algebras, (see [9]) since  $\lambda(xy) = \lambda(\chi(y)x)$ , we call  $\chi$  the generalized Nakayama automorphism for  $H$ .

*Remark 2.2.* For  $H$  finite dimensional,  $\chi(h) = \alpha(h_2)S^{-2}(h_1)$  and thus  $\epsilon \circ \chi$  is equal to  $\alpha$ , the distinguished grouplike in  $H^*$ . To see this, apply  $S^*$  to Lemma 1.1(ii), and recall that  $\lambda = S^*(\Lambda) \in \int_l^{H^*}$ . From (1.9) we obtain  $h \rightarrow \lambda = \alpha(h_2)\lambda \leftarrow S^{-2}(h_1)$ . Finally, apply  $\epsilon$  to  $\chi(h) = \alpha(h_2)S^{-2}(h_1)$  to see that  $\epsilon \circ \chi = \alpha$ .

The map  $\epsilon \circ \chi \in H^*$  is a convolution invertible algebra map with inverse  $\epsilon \circ \chi \circ S = \epsilon \circ \chi \circ S^{-1}$ . Remark 2.2 justifies the following terminology.

**Definition 2.3.** For  $H$  co-Frobenius and  $\chi$  the generalized Nakayama automorphism, the invertible algebra map  $\epsilon \circ \chi \in H^*$  will be denoted by  $\alpha$  and called the distinguished grouplike in  $H^*$ .

*Remark 2.4.* Suppose that  $\Lambda$  is a right integral for  $H$ , and  $\Omega$  is the algebra isomorphism defined by  $\Omega(h) \rightharpoonup \Lambda = \Lambda \leftharpoonup h$ . Then an easy computation shows that  $\Omega = S^{-1} \circ \chi \circ S = S \circ \chi \circ S^{-1}$  and so  $\epsilon \circ \Omega = \alpha^{-1}$ .

For  $H$  finite dimensional, the space of left integrals for  $H^*$  is invariant under the left adjoint action. This simple observation motivates the proof of the following generalization of the formula in Remark 2.2 to co-Frobenius Hopf algebras.

**Lemma 2.5.** *Let  $H$  be a co-Frobenius Hopf algebra,  $0 \neq \lambda \in \int_l^{H^*}$  and  $\alpha$  the distinguished grouplike element in  $H^*$ . Then for any  $h \in H$ ,*

$$\chi(h) = \alpha(h_2)S^{-2}(h_1).$$

*Proof.* We show first that for any  $h \in H$  the map  $S^2(h_2) \rightharpoonup \lambda \leftharpoonup S^{-1}(h_1)$  is a left integral. Indeed, for any  $h, h' \in H$  we have:

$$\begin{aligned} \langle S^2(h_2) \rightharpoonup \lambda \leftharpoonup S^{-1}(h_1), h'_2 h'_1 \rangle &= \langle \lambda, S^{-1}(h_1) h'_2 S^2(h_2) \rangle h'_1 \\ &= \langle \lambda, S^{-1}(h_1) h'_2 S^2(h_6) \rangle h_3 S^{-1}(h_2) h'_1 S^2(h_5) S(h_4) \\ (\lambda \in \int_l^{H^*}) \quad &= \langle \lambda, S^{-1}(h_1) h' S^2(h_4) \rangle h_2 S(h_3) \\ &= \langle S^2(h_2) \rightharpoonup \lambda \leftharpoonup S^{-1}(h_1), h' \rangle 1. \end{aligned}$$

Thus for any  $h \in H$  there is a scalar  $c_h \in k$  such that  $S^2(h_2) \rightharpoonup \lambda \leftharpoonup S^{-1}(h_1) = c_h \lambda$ , and this is equivalent to

$$\lambda \leftharpoonup \chi(S^2(h_2))S^{-1}(h_1) = c_h \lambda.$$

By (1.7), then  $\chi(S^2(h_2))S^{-1}(h_1) = c_h 1$ . Applying  $\epsilon$ , we obtain  $c_h = \alpha(S^2(h)) = \alpha(h)$  so that  $\chi(S^2(h_2))S^{-1}(h_1) = \alpha(h)1$ . Clearly, this is equivalent to  $\chi(S^2(h)) = \alpha(h_2)h_1$  and since  $S$  is bijective,  $\chi(h) = \alpha(h_2)S^{-2}(h_1)$ , for any  $h \in H$ .  $\square$

The next corollary generalizes [24, Corollary 5, p.599] to co-Frobenius Hopf algebras.

**Corollary 2.6.** *For  $H$  co-Frobenius with  $0 \neq \lambda \in \int_l^{H^*}$ , the following are equivalent.*

- i)  $H$  is involutory and  $\alpha = \epsilon$ , for  $\alpha$  the distinguished grouplike in  $H^*$ .
- ii) The integral  $\lambda$  is cocommutative.

*Proof.* Integrals in  $\int_l^{H^*}$  are cocommutative if and only if the generalized Nakayama automorphism  $\chi$  is the identity on  $H$ , so we need only show that  $\chi = id_H$  implies (i). But for  $h \in H$ , if  $h = \chi(h) = \alpha(h_2)S^{-2}(h_1)$ , then applying  $\epsilon$ , we obtain that  $\epsilon(h) = \alpha(h)$  and  $h = S^{-2}(h)$ .  $\square$

If  $H$  is finite dimensional, then applying Corollary 2.6 to the dual Hopf algebra  $H^*$ , we obtain the following statement.

**Corollary 2.7.** *Let  $H$  be a finite dimensional Hopf algebra and  $0 \neq t \in \int_l^H$ . Then  $t$  is cocommutative if and only if  $S^2 = id_H$  and  $g = 1$ .*

*Proof.* Apply Corollary 2.6 to  $H^*$ , identifying  $H$  with  $(H^*)^*$  and note that  $S^2 = id_H$  if and only if  $S^{*2} = id_{H^*}$ .  $\square$

Similar statements to Corollary 2.6 and Corollary 2.7 apply for right integrals.

We now prove the main result of this paper.

**Theorem 2.8.** *Let  $H$  be a co-Frobenius Hopf algebra, and  $\alpha, g$  the distinguished grouplike elements of  $H^*, H$ , respectively. Then for any  $h \in H$ ,*

$$S^4(h) = g(\alpha \rightharpoonup h \leftarrow \alpha^{-1})g^{-1}.$$

*Proof.* Let  $0 \neq \lambda \in \int_l^{H^*}$ . Then by (1.9),  $0 \neq \lambda' = \lambda \circ S = \lambda \leftarrow g^{-1} \in \int_l^{(H^{cop})^*}$ . Note that

$$\begin{aligned} h \rightharpoonup (\lambda \leftarrow g^{-1}) &= (h \rightharpoonup \lambda) \leftarrow g^{-1} \\ &= \lambda \leftarrow S^{-2}(h_1)\alpha(h_2)g^{-1} \text{ by Lemma 2.5} \\ &= (\lambda \leftarrow g^{-1}) \leftarrow (gS^{-2}(h_1)\alpha(h_2)g^{-1}), \end{aligned}$$

so that the generalized Nakayama automorphism  $\chi'$  for  $H^{cop}$  is given by  $\chi'(h) = gS^{-2}(h_1)\alpha(h_2)g^{-1}$ .

Note that  $\alpha'$ , the distinguished grouplike for  $H^{cop}$  is  $\epsilon \circ \chi' = \alpha$ . Also recall that the antipode for  $H^{cop}$  is  $S^{-1}$ . Now, from Lemma 2.5 for the co-Frobenius Hopf algebra  $H^{cop}$ , we have that

$$\chi'(h^{cop}) = \alpha'(h_2^{cop})S^2(h_1^{cop}) = \alpha(h_1)S^2(h_2).$$

Thus

$$\alpha(h_1)S^2(h_2) = gS^{-2}(h_1)\alpha(h_2)g^{-1}$$

and the statement follows immediately.  $\square$

The next corollary yields the analogue of [24, Theorem 3(b)] for the co-Frobenius case. In the finite dimensional case, this statement can be proved directly and then used together with Lemma 1.1(ii) or [24, Theorem 3(a)] to give a short proof of the  $S^4$  formula.

**Corollary 2.9.** *For any  $h \in H$ , we have that  $\chi(h) = \alpha(h_1)g^{-1}S^2(h_2)g$ .*

*Proof.* From Theorem 2.8,  $S^2(h) = g(\alpha \rightharpoonup S^{-2}(h) \leftarrow \alpha^{-1})g^{-1}$ , or, equivalently,  $S^{-2}(h) = g^{-1}(\alpha^{-1} \rightharpoonup S^2(h) \leftarrow \alpha)g$ . Substituting in the expression for  $\chi(h)$  from Lemma 2.5 yields the result.  $\square$

**Corollary 2.10.** *If  $H$  is co-Frobenius and  $\alpha$  and  $g$  have finite order, then the antipode  $S$  of  $H$  has finite order.*

The converse to Corollary 2.10 fails. In [2, Example 5.6], examples were given of infinite-dimensional co-Frobenius Hopf algebras for which  $S^4$  is the identity but the distinguished grouplike element in  $H$  has infinite order. The example below is based on [2, Example 5.6(i)] and provides, for any  $n$ , an example of an infinite dimensional co-Frobenius Hopf algebra  $\mathcal{H}$  with  $S^2$  of order  $n$ , distinguished grouplike in  $\mathcal{H}$  of infinite order, and distinguished grouplike in  $\mathcal{H}^*$  of order  $n$ .

**Example 2.11.** Let  $T_n$  be the Taft Hopf algebra of dimension  $n^2$ . Then  $T_n$  is generated as an algebra by  $x, c$  where  $c$  is grouplike and  $x$  is  $(1, c)$ -primitive, i.e.,  $\triangle(x) = x \otimes 1 + c \otimes x$ . Also  $c^n = 1$ ,  $x^n = 0$  and  $xc = qcx$  where  $q$  is a primitive  $n$ th root of unity. Let  $A = \langle a \rangle$  be an infinite cyclic group with identity  $e$  and let  $kA$  be the group algebra. Now, give  $T_n$  an  $A$ -grading by defining  $\deg(c^i x^j) = a^j$ . Then form the bicrossproduct  $\mathcal{H} = T_n \times kA$  with trivial weak action, cocycle and

dual cocycle. The Hopf algebra  $\mathcal{H}$  is equal to the tensor product  $T_n \otimes kA$  as an algebra but has comultiplication defined by  $\Delta(c^i x^j \otimes a^k) = \sum_{0 \leq t \leq j} \binom{j}{t}_q (c^i c^t x^{j-t} \otimes a^t a^k) \otimes (c^i x^t \otimes a^k)$ . By [2, Theorem 5.1],  $S(c^i x^j \otimes a^k) = S_{T_n}(c^i x^j) \otimes S_{kA}(a^{j+k})$ . Thus the order of  $S_{\mathcal{H}}$  is  $2n$  since the antipode for  $T_n$  has order  $2n$ , and the antipode for a group algebra has order 2.

From the proof of [2, Proposition 5.2],  $\Lambda = p_{x^{n-1}} \otimes p_e$  is a nonzero right integral for  $\mathcal{H}$  where  $p_h(l) = \delta_{h,l}$ . We compute the order of the distinguished grouplike in  $\mathcal{H}$  using this right integral. Clearly  $(c^i x^j \otimes a^k) \rightharpoonup \Lambda$  is nonzero only on elements of the form  $c^{n-i} x^{n-j-1} \otimes a^{-k}$ . But we have that  $q^i(c^{n-i} x^{n-j-1} \otimes a^{-k})(c^i x^j \otimes a^k) = (c^i x^j \otimes a^k)(c^{n-i} x^{n-j-1} \otimes a^{-k})$ . Since  $q$  is a primitive  $n$ th root of unity, then the distinguished grouplike in  $\mathcal{H}^*$  has order  $n$ . By [2, Proposition 5.4], the distinguished grouplike in  $\mathcal{H}$  is  $c^{n-1} \otimes a^{n-1}$ , which has infinite order.  $\square$

In general, for  $H$  co-Frobenius, the order of  $S$  need not be finite. For example, let  $\text{char}(k) \neq 2$ ,  $q \in k$  not a root of unity, and let  $H = k[SL_q(2)]$ , the coordinate ring of  $SL_q(2)$  [29]. Then  $H$  is a cosemisimple Hopf algebra whose antipode has infinite order. (In fact, by [2, Proposition 1.2] no power of the antipode of  $H$  is even inner.) The distinguished grouplike in  $H$  is 1. In the next example, we compute the map  $\chi$  and the distinguished grouplike element  $\alpha \in H^*$  for  $H = k[SL_q(2)]$ . We use the notations of [29].

**Example 2.12.** Recall that  $H = k[SL_q(2)]$  is the Hopf algebra generated as an algebra by  $a, b, c, d$  with relations

$$ba = qab, ca = qac, db = qbd, dc = qcd, bc = cb, da - qbc = 1, ad - q^{-1}bc = 1.$$

The coalgebra structure  $\Delta, \varepsilon$  and the antipode  $S$  are given by

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c, & \Delta(b) &= a \otimes b + b \otimes d, \\ \Delta(c) &= c \otimes a + d \otimes c, & \Delta(d) &= c \otimes b + d \otimes d, \\ \varepsilon(a) &= \varepsilon(d) = 1, & \varepsilon(b) &= \varepsilon(c) = 0, \\ S(a) &= d, & S(b) &= -qb, & S(c) &= -q^{-1}c, & S(d) &= a. \end{aligned}$$

Elements of the form  $a^i b^j c^k d^l$  with  $i, j, k, l \in \mathbb{N}$  such that either  $i = 0$  or  $l = 0$  form a basis for  $H$ . A left integral is given by  $\lambda \in H^*$  defined by

$$\lambda((bc)^n) = (-1)^n / [n+1] \text{ where } [i] = (q^i - q^{-i}) / (q - q^{-1})$$

and  $\lambda$  maps all other basis elements to 0. Since  $H$  is cosemisimple,  $g = 1$ .

From the formula for  $S^4$  in Theorem 2.8, letting  $h = a$ , we obtain that  $\alpha(c) = \alpha(b) = 0$ , and letting  $h = b$ , we obtain that  $\alpha(d) = q^4 \alpha(a)$ . To find  $\alpha(a)$ , we compute

$$\begin{aligned} \langle a \rightharpoonup \lambda, d \rangle &= \lambda(da) = \lambda(qbc + 1) = q\lambda(bc) + \lambda(1) \\ &= -\frac{q(q - q^{-1})}{q^2 - q^{-2}} + 1 = \frac{1}{q^2 + 1}, \end{aligned}$$

and from Lemma 2.5 we have

$$\begin{aligned} \langle \lambda \leftarrow \chi(a), d \rangle &= \alpha(a)\lambda(ad) = \alpha(a)\lambda(q^{-1}bc + 1) \\ &= \alpha(a) \left[ -\frac{q^{-1}(q - q^{-1})}{q^2 - q^{-2}} + 1 \right] = \alpha(a) \frac{q^2}{q^2 + 1}. \end{aligned}$$

Thus  $\alpha(a) = q^{-2}$ . Summarizing, we have that

$$\begin{aligned} \alpha(a) &= q^{-2}, \quad \alpha(b) = \alpha(c) = 0, \quad \alpha(d) = q^2; \\ \chi(a) &= q^{-2}a, \quad \chi(b) = b, \quad \chi(c) = c, \quad \chi(d) = q^2d. \quad \square \end{aligned}$$

### 3. INTEGRALS AND THE SQUARE OF THE ANTIPODE

Kaplansky's fifth conjecture states that a finite dimensional semisimple or cosemisimple Hopf algebra  $H$  is involutory, i.e., that  $S^2 = id_H$ . Over time, this conjecture has been split into two problems. The first is whether a semisimple Hopf algebra is cosemisimple and the second is whether a semisimple cosemisimple Hopf algebra is involutory. The most substantial progress on the first problem was made by Larson and Radford in [16, 17]. They proved that over a field of characteristic zero a finite dimensional Hopf algebra  $H$  is semisimple if and only if  $H$  is cosemisimple, if and only if  $H$  is involutory. Recently, streamlined proofs using the Frobenius algebra approach appeared in [18] for these results. Larson and Radford also proved that in characteristic  $p$  sufficiently large a semisimple cosemisimple Hopf algebra is involutory. Ten years later, Etingof and Gelaki [11] completed the solution for the second problem. In fact, using the results of Larson and Radford on the one hand and a lifting theorem on the other hand, they proved the following result.

**Theorem 3.1.** [11] *Let  $H$  be a finite dimensional Hopf algebra over  $k$ . Then  $H$  is semisimple and cosemisimple if and only if  $S^2 = id_H$  and  $\dim(H)1 \neq 0$  in  $k$ .*

In this section, we explore equivalent conditions to  $H$  involutory for  $H$  co-Frobenius, not necessarily finite dimensional, in terms of integrals.

*Remark 3.2.* Let  $H$  be co-Frobenius with  $g = 1$ . Let  $\lambda \in \int_l^{H^*} = \int_r^{H^*}$ ;  $\lambda \circ S = \lambda$ . Then the following conditions are equivalent:

- (i)  $H$  is involutory.
- (ii) The bilinear form  $B(x, y) = \lambda(xS(y))$  is symmetric.

For if  $S^2 = id_H$ , then  $\lambda(xS(y)) = \lambda \circ S(xS(y)) = \lambda(S^2(y)S(x)) = \lambda(yS(x))$ . Conversely,  $\lambda(xS(y)) = \lambda(yS(x)) = \lambda \circ S(yS(x)) = \lambda(S^2(x)S(y))$ , so that  $\lambda \leftarrow x = \lambda \leftarrow S^2(x)$  for all  $x$ , and thus  $S^2 = id_H$ .

In [15, Corollary 3.6], Larson proved that over an algebraically closed field, a cosemisimple Hopf algebra  $H$  is involutory if and only if  $\lambda(h_2S(h_1)) = \epsilon(h)$  for some  $\lambda \in \int_l^{H^*}$ , all  $h \in H$ . The proposition below generalizes this result.

**Proposition 3.3.** *For  $H$  co-Frobenius over a field  $k$ , the following are equivalent:*

- i)  $H$  is cosemisimple and involutory.
- ii)  $H^*$  has a non-zero left or right integral  $\lambda$  such that  $\lambda(S(h_2)h_1) = \epsilon(h)$ , for all  $h \in H$ .
- iii)  $H^*$  has a non-zero left or right integral  $\lambda$  such that  $\lambda(h_2S(h_1)) = \epsilon(h)$  for all  $h \in H$ .
- iv)  $H$  is cosemisimple and there is a cocommutative integral for  $H$  in  $H^*$ .

*Proof.* We show first that (i),(ii) and (iii) are equivalent. The proof that (i) implies (ii) is the proof that (a) implies (c) in [15, Corollary 3.6]. The implications (i) implies (ii) and (i) implies (iii) also follow immediately from Remark 3.2 and the



fact that since  $H$  is cosemisimple, a left and right integral  $\lambda$  may be chosen so that  $\lambda(1) = 1$ .

Now suppose (ii) holds and  $\lambda$  is a left or right integral with  $\lambda(S(h_2)h_1) = \varepsilon(h)$ , for all  $h \in H$ . Then  $\lambda(1) = 1$  and so  $H$  is cosemisimple by the Dual Maschke Theorem and  $\lambda$  is both a left and right integral. To see that  $H$  is involutory, we compute, for any  $h \in H$ ,

$$\begin{aligned} h = h \cdot 1 &= \lambda(S(h_3)h_2)h_1 \\ &= \lambda(S(h_3)h_2)S^2(h_5)S(h_4)h_1 \\ &= \lambda(S(h_2)h_1)S^2(h_3) \\ &= S^2(h). \end{aligned}$$

Similarly, if (iii) holds, then  $H$  is cosemisimple, and to see that  $H$  is involutory, compute  $h = \lambda(h_2S(h_1))h_3 = \lambda(h_4S(h_3))h_5S(h_2)S^2(h_1)$ . Thus each of (ii) and (iii) implies (i).

By Corollary 2.6, it is clear that (iv) implies (i). Conversely, suppose that the equivalent conditions (i)-(iii) hold. Note that then the integral in both (ii) and (iii) is the unique integral  $\lambda = \lambda \circ S$  such that  $\lambda(1) = 1$ . We show that this implies that  $\alpha = \epsilon$ . For  $h \in H$  and  $\lambda = \lambda \circ S$  an integral in  $H^*$  such that  $\lambda(1) = 1$ , note that

$$\begin{aligned} \epsilon(h) = \lambda(h_2S(h_1)) &= \langle S(h_1) \rightharpoonup \lambda \leftarrow h_2, 1 \rangle \\ &= \langle \lambda \leftarrow \alpha(S(h_1))S^{-2}(S(h_2))h_3, 1 \rangle \\ &= \alpha^{-1}(h_1) \langle \lambda, S^{-1}(h_2)h_3 \rangle \\ &= \alpha^{-1}(h_1)\lambda(S(h_3)h_2) \\ &= \alpha^{-1}(h). \end{aligned}$$

Thus  $\alpha = \epsilon$  and (iv) follows from Corollary 2.6.  $\square$

Note that if  $H$  is cosemisimple, or more generally if  $H$  is symmetric as a coalgebra [5], then  $S^2$  is “coinver”, meaning that  $S^2(h) = v(h_1)h_2v^{-1}(h_3)$  for some invertible  $v \in H^*$ . Then  $S^2 = id_H$  if and only if  $v * id_H = id_H * v$ .

*Remark 3.4.* The equivalent conditions for  $S^2 = id_H$  for  $H$  co-Frobenius may remind the reader of analogous conditions in the quasi-triangular and coquasi-triangular case. It is a well-known theorem of Drinfel'd [6] (also proved by Radford [23]) that for  $(H, R = R^{(1)} \otimes R^{(2)})$  quasi-triangular, then  $S^2$  is an inner automorphism of  $H$ , i.e.,  $S^2(h) = uhu^{-1}$  where  $u = S(R^{(2)})R^{(1)}$ .

Dual results hold for  $H$  coquasi-triangular [25, Lemma 3.3.2], [8, Theorem 1.3]. Let  $(H, \beta)$  be a coquasi-triangular Hopf algebra and  $u \in H^*$  be defined by  $u(h) = \beta(h_2, S(h_1))$ . Then  $u$  is a unit in  $H^*$  and  $S^2(h) = u(h_1)h_2u^{-1}(h_3)$ . These results were generalized to weakly coquasi-triangular Hopf algebras in [1].

Now we restrict to the finite dimensional case.

**Corollary 3.5.** *If  $H$  is finite dimensional, then there exists a non-zero left or right integral  $t$  in  $H$  such that  $S(t_2)t_1 = 1$  or such that  $t_2S(t_1) = 1$  if and only if  $H$  is semisimple and involutory.*

Now, for  $H$  finite dimensional, we have a list of equivalent statements to  $H$  involutory.

**Theorem 3.6.** *Let  $H$  be a finite dimensional Hopf algebra over a field  $k$  such that  $\dim(H)1 \neq 0$  in  $k$ . Then the following are equivalent:*

- i)  $H$  is semisimple and cosemisimple.
- ii)  $H$  is involutory.
- iii)  $H$  has a non-zero left or right cocommutative integral  $t \in H$ .
- iv) There exists  $0 \neq t \in \int_l^H$  or  $0 \neq t \in \int_r^H$  such that  $S(t_2)t_1 = 1$  or  $t_2S(t_1) = 1$ .
- v)  $H^*$  has a non-zero left or right cocommutative integral  $\lambda \in H^*$ .
- vi)  $H^*$  has a non-zero left or right integral  $\lambda$  such that  $\lambda(h_2S(h_1)) = \varepsilon(h)$  or  $\lambda(S(h_2)h_1) = \varepsilon(h)$ , for all  $h \in H$ .

*Proof.* The equivalence of i) and ii) follows from Theorem 3.1. By Corollary 2.7, with the assumption that  $\dim(H)1 \neq 0$ , i) and ii) are equivalent to the existence of a cocommutative integral in  $H$ , i.e., to iii). Corollary 3.5 implies that i) and ii) are equivalent to iv). Thus the first four conditions are equivalent.

The equivalence of the remaining conditions is proved similarly using Corollary 2.6, Theorem 3.1 and Proposition 3.3.  $\square$

*Remark 3.7.* The condition  $\dim(H)1 \neq 0$  in  $k$  in Theorem 3.6 is essential. Let  $p$  be prime,  $\text{char}(k) = p$  and  $C_p = \langle a \rangle$  a cyclic group of order  $p$ . Then  $H = kC_p$  is an involutory cosemisimple Hopf algebra for which  $\dim(H) = 0$  in  $k$ . Moreover,  $t = \sum_{n=0}^{p-1} a^n$  is a non-zero left and right cocommutative integral for  $H$  and  $p_e \in H^*$  defined as in Example 2.11 is a non-zero left and right cocommutative integral for  $H^*$  and satisfies the conditions in Theorem 3.6 vi) by Proposition 3.3. But  $H$  is not semisimple and  $S(t_2)t_1 = t_2S(t_1) = 0$ . So here ii), iii), v) and vi) in Theorem 3.6 hold, while i) and iv) do not. Note that in this example, the grouplikes  $g$  and  $\alpha$  are trivial,  $\lambda(1) = 1$  but  $\epsilon(t) = 0$ .

*Remark 3.8.* If  $\text{char}(k) = 0$  then by the results of Larson and Radford mentioned above,  $H$  semisimple is equivalent to  $H$  cosemisimple and each of these conditions is equivalent to statements (iii) through (vi) in Theorem 3.6.

We end this paper by explicitly constructing the cocommutative integrals for  $H$  and  $H^*$  when the finite dimensional Hopf algebra  $H$  is both semisimple and cosemisimple or, equivalently, when  $H$  is involutory and  $\dim(H) \neq 0$  in  $k$ .

To this end, consider  $\{e_i\}$  a basis in  $H$  with dual basis  $\{e^i\}$ . We know from [24] that the element  $r \in H$  defined by

$$p(r) = \text{Tr}(l(p) \circ S^2), \quad \forall p \in H^*$$

is a non-zero right integral for  $H$  if and only if  $H$  is cosemisimple, where we denoted by  $l(p)$  the endomorphism of  $H$  defined by  $l(p)(h) = p(h_2)h_1$ , for any  $h \in H$ , and by  $\text{Tr}(l(p) \circ S^2)$  the trace of the  $H$ -endomorphism  $l(p) \circ S^2$ . So we have

$$\begin{aligned} p(r) &= \text{Tr}(l(p) \circ S^2) = \text{Tr}(S^2 \circ l(p)) \\ &= \sum_{i=1}^n \langle e^i, S^2(l(p)(e_i)) \rangle = \sum_{i=1}^n \langle e^i, S^2((e_i)_1) \rangle p((e_i)_2), \end{aligned}$$

for all  $p \in H^*$ , and thus

$$r = \sum_{i=1}^n \langle e^i, S^2((e_i)_1) \rangle (e_i)_2.$$

Replacing  $H$  by  $H^{\text{op}, \text{cop}}$  we get that

$$t = \sum_{i=1}^n \langle e^i, S^2((e_i)_2) \rangle (e_i)_1$$

is a non-zero left integral for  $H$  if and only if  $H$  is cosemisimple. Note that  $\varepsilon(r) = \varepsilon(t) = \text{Tr}(S^2)$ . In particular, if  $H$  is also semisimple then  $r = t$ .

Dually, we have that  $\lambda, \Lambda \in H^*$  defined by

$$\lambda = \sum_{i=1}^n S^2(e_i) \rightharpoonup e^i \text{ and } \Lambda = \sum_{i=1}^n e^i \leftharpoonup S^2(e_i)$$

are non-zero left, respectively right, integrals for  $H^*$  if and only if  $H$  is semisimple. Clearly,  $\lambda(1) = \Lambda(1) = \text{Tr}(S^2)$ , so if  $H$  is also cosemisimple then  $\lambda = \Lambda$ .

Now, Theorem 3.6 and the uniqueness of integrals for  $H$  and  $H^*$  imply:

**Corollary 3.9.** *Let  $H$  be a finite dimensional semisimple cosemisimple Hopf algebra, and  $\{e_i\}$  a basis in  $H$  with dual basis  $\{e^i\}$ . Then*

- i)  $0 \neq t = \sum_{i=1}^n \langle e^i, (e_i)_1 \rangle (e_i)_2 = \sum_{i=1}^n \langle e^i, (e_i)_2 \rangle (e_i)_1$  is a cocommutative integral;
- ii)  $0 \neq \lambda = \sum_{i=1}^n e_i \rightharpoonup e^i = \sum_{i=1}^n e^i \leftharpoonup e_i$  is a cocommutative integral.

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